Representations of the quantum matrix algebra $M_{q,p}(2)$

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Abstract. It is shown that the finite dimensional irreducible representations of the quantum matrix algebra $M_{q,p}(2)$ (the coordinate ring of $GL_{q,p}(2)$) exist only when both q and p are roots of unity. In this case the space of states has either the topology of a torus or a cylinder, which may be thought of as generalizations of cyclic representations.

1. Introduction

The representation theory of quantized universal enveloping algebras [1–5] has been tensively studied by many authors [6–10] and many beautiful results have been obtained. Among these are the cyclic representations which occur when q is a root of unity. However, the representation theory of the dual objects, that is the quantization of the algebra of functions on the group (quantum matrix algebras) has not been systematically studied.

Only a few concrete representations exist [11, 12]. The first attempt towards such a goal has been reported in [13], where the irreducible finite dimensional representations of $M_q(2)$ were classified and it was shown that such irreducible representations exist only when q is a root of unity. These representations were either cyclic or highest weight.

Although the representation theory of a multiparametric quantum enveloping algebra is trivial (since the extra parameters appear only in the coproducts in the form of a twisting [4]), the corresponding task for the multiparametric quantum matrix group is far from straightforward, since the extra parameters now appear in the algebra itself. As we will see this will lead to quite new features in the representations of the quantum matrix algebra.

In this paper we study the representations of $M_{q,p}(2)$ (the coordinate ring of $GL_{q,p}(2)$) and classify its finite dimensional irreducible representations. Our main results are the following:

(i) Finite dimensional irreducible representations exist only when both q and p are roots of unity.

(ii) The space of states has the topology of a torus or that of a cylinder, depending on the value of some parameters which define the presentation.

(iii) When none of the parameters q or p is a root of unity, the states occupy the sites of an infinite rectangular lattice.

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(iv) When q is not a root of unity, but p is, the states occupy the sites of an infinitely long cylinder.

2. The quantum matrix algebra $GL_{q,p}(2)$

This algebra [15] is generated by the entries of a matrix

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and subject to the relations:

$$RT_1T_2 = T_2T_1R \tag{1}$$

where R is a two parametric solution of the Yang–Baxter equation:

$$R = \begin{pmatrix} q & & & \\ & p & & \\ & q-q^{-1} & p^{-1} & \\ & & & q \end{pmatrix}. \tag{2}$$

The relations which follow from (1) and (2) are

$$ab = qp\, ba \qquad bd = \frac{q}{p} db$$

$$ac = \frac{q}{p} ca \qquad cd = qp\, dc \tag{3}$$

$$bc = \frac{1}{p^2} cb \qquad ad - da = p(q - q^{-1})\, bc.$$

Compared with $GL_q(2)$ this quantum group has two particular features which make its representation theory quite different. The first is that the generators b and c no longer commute and hence one cannot build the representation space from common eigenvectors of b and c as in the work of [13]. The second is that the quantum determinant $D = ad - qp\, bc$ is not central, but satisfies the relations:

$$Da = aD \qquad Db = p^2 bD$$

$$Dd = dD \qquad Dc = p^{-2} cD. \tag{4}$$

The following relations can also be obtained by repeated application of (3)

$$d^n d - d d^n = qp(1 - q^{-2n}) d^{n-1} bc$$

$$ad^n - d^n a = \frac{p}{q}(q^{2n} - 1) d^{n-1} bc \tag{5}$$

when $q^n = p^n = 1$ one sees from (3) and (5) that a^n, b^n, c^n and d^n are central.

In [13] both the commutativity of b and c and the centrality of the quantum determinant have been used to a large extent. In our case when the above facts are no longer true, we must proceed in a different way.

Denote the product of b and c by $M(M=bc)$. Then from (3) we find

$$Ma=q^{-2}aM \qquad Mb=p^{2}bM$$
$$Mc=p^{-2}cM \qquad Md=q^{2}dM. \tag{6}$$

It is clear from (4) that the operators M and D are commuting:

$$MD=DM. \tag{7}$$

We will use these commuting operators in the following sections to build up the irreps. of $M_{q,p}(2)$.

3. Finite dimensional irreducible representations

Following [13] we call those $M_{q,p}(2)$ modules in which one or more of the generators identically vanish, trivial modules. In these cases the representation reduces to that of a simpler algebra. Clearly the interesting representations are non-trivial ones to which we restrict ourselves in the rest of this paper. The following lemma [13] establishes the condition for non-triviality of the representation.

Lemma 1. Let V be a vector space. Then an irreducible representation $\rho : M_{q,p}(2) \rightarrow \mathrm{End}(V)$ is trivial if b or c have zero eigenvalue in their spectrum.

Proof. We follow a slightly different line of reasoning which is simpler than that of [13]. Without loss of generality, lets assume that $K_b \equiv \mathrm{Ker}\, b \neq \{0\}$, then since K_b is a subspace of V, we can choose a basis for it like: $\{e_1, ... e_m\}$. From (3) we see that aK_b, cK_b and dK_b are all subspaces of K_b. Therefore the vectors $\{e_i\}$ transform among themselves under the action of a, b, c and d and hence K_b is an invariant subspace of V. Since the representation is irreducible $K_b = V$. Therefore $bK_b = bV = 0$ and the representation is trivial. Hereafter we assume that $K_b = K_c = \{0\}$.

Lemma 2. A finite dimensional irreducible $M_{q,p}(2)$ module exists only when both q and p are roots of unity.

Proof. Let v_0 be a common eigenvector of M and D.

$$Mv_0 = \mu v_0 \qquad Dv_0 = \lambda v_0. \tag{8}$$

Then from (3), (4) and (6) one sees that the string of states $v_n \equiv d^n v_0$ satisfy:

$$Mv_n = q^{2n} \mu v_n \qquad Dv_n = \lambda v_n. \tag{9}$$

For the parameter q we adopt the reasoning of [13]. To have finite dimensional representations one must have $d^l v_0 = 0$ for some l while all the vectors v_n for $n < l$ are independent. Consider the string of states $a^m u_0$ where $u_0 = v_{l-1}$. Again one must have $a^{l'} u_0 = 0$ but $a^{l'-1} u_0 \neq 0$. Then one will have

$$0 = da^{l'} u_0 = qp(q^{-2l'} - 1) \mu a^{l'-1} u_0$$

which means that q must be a root of unity. One the other hand, suppose that p is not a root of unity; then the string of states $u_n = b^n v_0$ satisfy

$$Mu_n = p^{2n} \mu \, u_n \qquad Du_n = p^{2n} \lambda u_n.$$

If p is not a root of unity all the above states will be independent and the representation cannot be finite.

Hereafter we set $q^r = p^r = 1$ (note: q and p may be different roots of unity, i.e $q^{r_1} = p^{r_2} = 1$. We set r to be the least common multiplier of r_1 and r_2).

In this case a^r, b^r, c^r and d^r are central and on V we set them equal to η_a, η_b, η_c and η_d respectively. Clearly η_b and η_c are both different from zero, otherwise K_b and K_c will have non-zero elements.

We now classify the finite dimensional irreducible representations of $M_{q,p}(2)$ which fall into three classes depending on the values of η_a and η_d.

3.1. Toroidal representations $(\eta_a \neq 0 \neq \eta_d)$

We denote the vector v_0 introduced in (8) by $|0, 0\rangle$ and consider the lattice of states W (figure 1):

$$W = \{ |l, n\rangle \equiv b^l d^n |0, 0\rangle \qquad 0 \leqslant l, n \leqslant r-1 \}. \tag{10}$$

These states are the common eigenvectors of M and D (see (3)–(6)).

$$M|l, n\rangle = p^{2l} q^{2n} \mu |l, n\rangle \qquad D|l, n\rangle = p^{2l} \lambda |l, n\rangle. \tag{11}$$

We define the action of a and c on the state $|0, 0\rangle$ as follows:

$$a|0, 0\rangle = \alpha_0 |0, r-1\rangle \qquad c|0, 0\rangle = \gamma_0 |r-1, 0\rangle. \tag{12}$$

Then we have:

Theorem 3. The following defines an irreducible representation of $M_{q,p}(2)$.

(i) $b|l, n\rangle = |l+1, n\rangle \qquad b|r-1, n\rangle = \eta_b |0, n\rangle$

(ii) $d|l, n\rangle = \left(\dfrac{p}{q}\right)^l |l, n+1\rangle \qquad d|l, r-1\rangle = \left(\dfrac{p}{q}\right)_l \eta_d |l, 0\rangle$

(iii) $c|l, n\rangle = p^{2l} q^{2n} \eta_b \gamma_0 |l-1, n\rangle \qquad c|0, n\rangle = q^{2n} \gamma_0 |r-1, n\rangle$

(iv) $a|l, n\rangle = (qp)^l \left(\alpha_0 \eta_d + \dfrac{p}{q} (q^{2n}-1)\gamma_0 \eta_b \right) |l, n-1\rangle \qquad a|l, 0\rangle = (qp)^l \alpha_0 |l, r-1\rangle.$

Proof. (i) and (ii) are obvious. We give an explicit verification of (iii). (iv) is obtained by straightforward manipulations. Acting with c on the state $|l, n\rangle$ and using the commutation relations (3) we find

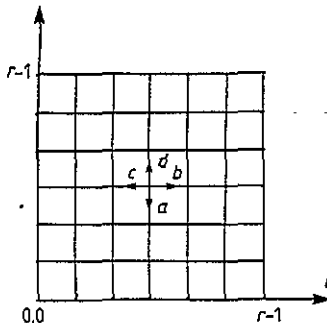$$c|l, n\rangle = c b^l d^n |0, 0\rangle = p^{2l} (qp)^n b^l d^n c |0, 0\rangle.$$

Figure 1. The lattice of states W.

Using (12) and definition of states (10) we arrive at:

$$c|l, n\rangle = p^{2l}(qp)^n b^l d^n \gamma_0 |r-1, 0\rangle = p^{2l}(qp)^n \gamma_0 b^l d^n b^{r-1}|0, 0\rangle.$$

Again using commutation relations and the fact that $b^r = \eta_b$ we finally find

$$c|l, n\rangle = p^{2l}(qp)^n \gamma_0 \left(\frac{p}{q}\right)^{n(r-1)} b^{l-1}d^n|0, 0\rangle = p^{2l}q^{2n}\eta_b\gamma_0|l-1, n\rangle.$$

The second part of (iii) is proved similarly:

$$c|0, n\rangle = cd^n|0, 0\rangle = (qp)^n d^n c|0, 0\rangle$$
$$= (qp)^n d^n \gamma_0|r-1, 0\rangle = (qp)^n \gamma_0 d^n b^{r-1}|0, 0\rangle$$
$$= (qp)^n \gamma_0 \left(\frac{p}{q}\right)^{r(r-1)} b^{r-1}d^n|0, 0\rangle = q^{2n}\gamma_0|r-1, n\rangle.$$
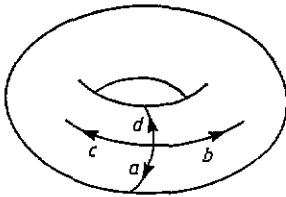

Figure 2. Toroidal representation.

It is interesting to note that the space of states has the topology of a torus (see figure 2 where the actions of all the generators are shown graphically).

The dimension of this representation is r^2. To prove that it is the only irreducible representation in this case, we note that the dimension of V cannot be greater than r^2 since otherwise the above lattice of states, which is based on a single common eigenvector of M and D, will provide an invariant subspace which contradicts the irreducibility of the representation. The dimension of V cannot be less that r^2 either, since then one of the strings of states $d^n|0, 0\rangle$ or $b^n|0, 0\rangle$ must terminate for some value of n less than r (i.e. $d^n|0, 0 = 0, n < r$) This then means that

$$\eta_d|0, 0\rangle = d^{r-n}(d^n|0, 0\rangle) = 0$$

which contradicts the assumption of $\eta_d \neq 0$.

Remark. The parameters a_0 γ_0 λ and μ are not independent of η_a, η_b, η_c and η_d. The following relations exist among them:

$$\eta_c = \gamma_0^r \eta_b^{r-1} \qquad \mu = \gamma_0 \eta_b \tag{13}$$

$$\lambda = a_0 \eta_d - \frac{p}{q}\mu \qquad \eta_a = a_0 \prod_{i=1}^{r-1}\left(\lambda + \frac{p}{q}\mu q^{2i}\right). \tag{14}$$

The proof of these formulae is given in the appendix. We now turn to the second kind of representations.

3.2. Cylindrical representations ($\eta_a = \eta_d = 0$)

In this case the representation in theorem 3 is modified as follows:

$$d|l, r-1\rangle = 0 \qquad a|l, 0\rangle = 0 \tag{15}$$

$$a|l, n\rangle = (qp)^l \left(\frac{p}{q} (q^{2n} - 1) \gamma_0 \eta_b \right) |l, n-1\rangle. \tag{16}$$

All other relations remain intact. This kind of representation may be called a cylindrical representation and may be thought of as a truncated form of the toroidal representation (figure 3).
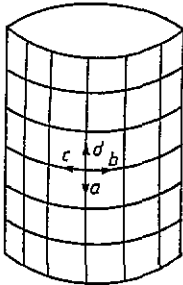

Figure 3. Cylindrical representation.

Where only one of the parameters (say η_a) is zero, the relations (15) and (16) are modified as follows:

$$d|l, r-1\rangle = \eta_a |l, 0\rangle \qquad a|l, 0\rangle = 0 \tag{17}$$

$$a|l, n\rangle = (qp)^l \left(\frac{p}{q} (q^{2n} - 1) \gamma_0 \eta_0 \right) |l, n-1\rangle. \tag{18}$$

This then may be thought of as a semitoroidal representation in which d traverses completely one of the cycles of the torus while a does not.

4. Infinite dimensional representations

In theorem 3 one can relax the conditions on the right-hand side. Then one can check easily that the left-hand side equations define an infinite dimensional representation of $M_{q,p}(2)$ on the two-dimensional lattice: $W = \{|l, n\rangle - \infty < l, n < \infty\}$

$$b|l, n\rangle = |l+1, n\rangle$$

$$d|l, n\rangle = \left(\frac{p}{q} \right)^l |l, n+1\rangle$$

$$c|l, n\rangle = p^{2l} q^{2l} \mu |l-1, n\rangle \tag{19}$$

$$a|l, n\rangle = (qp)^l \left(\lambda + \left(\frac{p}{q} \mu \right) q^{2n} \right) |l, n-1\rangle.$$

Note. The states of this representation are not necessarily built up on a vacuum (i.e. $|l, n\rangle \neq b^l d^n |0, 0\rangle$).

If both q and p are roots of unity ($q^r = p^r = 1$) then one can consistently identify the states as follows:

$$|l, n\rangle \equiv |l+r, n\rangle \qquad\qquad |l, n\rangle \equiv |l, n+r\rangle.$$

Representation (19) will induce a toroidal representation on the equivalence class of these states.

If only p is a root of unity, then one can consistently identify the states as follows:

$$|l+r, n\rangle \equiv q^{nr}|l, n\rangle$$

Equations (19) then induce a cylindrical representation on the equivalence of these states. By setting $\lambda = -(p/q)\mu$ one can also obtain a lowest weight module.

5. Discussion

In the classical limit ($q = p = 1$), the coordinate ring of $GL_{q,P}(2)$ degenerates into a free Abelian algebra whose irreducible finite dimensional representations are one dimensional wher the generators a, b, c and d are represented by the pure numbers η_a, η_b, η_c, and η_d, respectively. This limit is obtained from the representations in this paper by noting that when $r = 1$, the lattice of states in figure 1 has only one single state, namely $|0, 0\rangle$ and from theorem 1 we have

$$a|0, 0\rangle = \alpha_0|0, 0\rangle \qquad\qquad b|0, 0\rangle = \eta_b|0, 0\rangle$$

$$c|0, 0\rangle = \gamma_0|0, 0\rangle \qquad\qquad d|0. 0\rangle = \eta_d|0, 0\rangle.$$

From (14) and (15) we also obtain $\gamma_0 = \eta_c$ and $\alpha_0 = \eta_a$ which proves the assertion.

What appears to be very interesting about the representation theory of quantum matrix algebras compared with those of the quantized universal enveloping algebras is that in the latter case the classical limit is a lie algebra and one can use the decomposition of the root space of the lie algebra into the Cartan subalgebra and positive and negative root system for building up the representation. Recall that this decomposition remains essentially intact in the process of quantization. This then leads to a parallelism between the representation theory in the deformed and the undeformed case. However, for the case of quantum matrix algebras such a decomposition and the resulting parallelism do not exist. One expects that completely new features arise in their representation theory (see, for example [16, 17]).

Appendix. Proof of equation (13) and (14)

Repeated use of theorem 3 gives the following:

$$c^{r-1}|r-1, 0\rangle = (\eta_b\gamma_0)^{r-1}|0, 0\rangle.$$

Acting by c on both sides we obtain

$$\eta_c|r-1, 0\rangle = (\eta_b\gamma_0)^{r-1}\gamma_0|r-1, 0\rangle$$

where we have used (12). Comparison of both sides gives the first relation of (13). To obtain the second relation of (13) we note that

$$\mu|0, 0\rangle = M|0, 0\rangle = bc|0, 0\rangle = b(\gamma_0|r-1, 0\rangle) = \gamma_0\eta_b|0, 0\rangle.$$

To prove the first relation of (14) we act on the state $|0, 0\rangle$ with $D = ad - qpM$ and use theorem 3. Finally we note that

$$a^{r-1}|0, r-1\rangle = \prod_{i=1}^{r-1}\left(\lambda + \left(\frac{p}{q}\mu\right)q^{2i}\right)|0, 0\rangle.$$

Acting on both sides with a and using (12) gives (14).

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